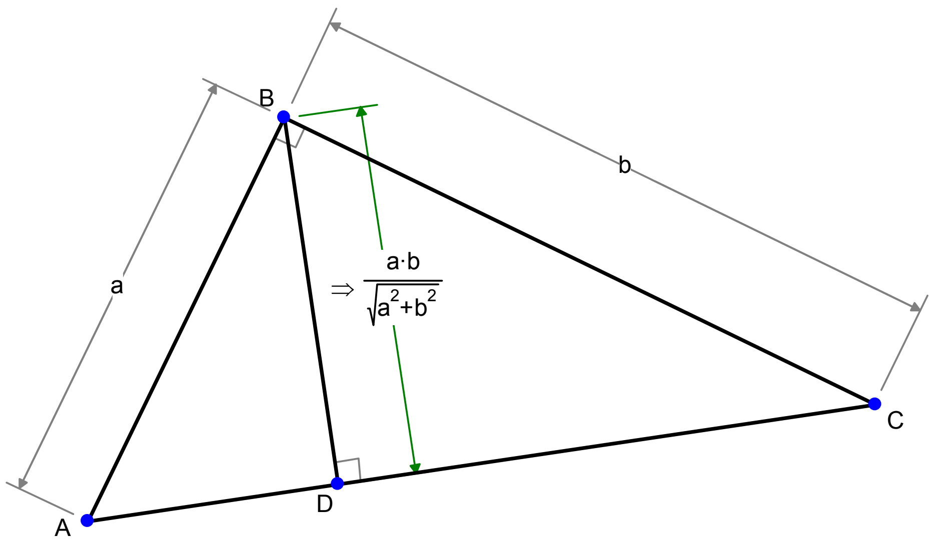
Insight with Geometry Expressions

Introduction

Geometry Expressions automatically generates algebraic expressions from geometric figures. For example in the diagram below, the user has specified that the triangle is right and has short sides length a and b. The system has calculated an expression for the length of the altitude:



Can a system which does geometry be used in teaching geometry? I'd suggest the answer is yes. And this article is my attempt to justify that answer by example.

In 1981, I was supplementing my income as a research assistant by tutoring mathematics. One of my students was studying for his "City and Guild" exam in electronics. The examination's contents had not been updated in a decade, and part of the exam was to perform multiplication and power calculations using log tables. Mathematicians of my age or older will remember the peculiar arithmetic used with log tables where numbers less than 1 were involved. By 1982 the advent of inexpensive pocket calculators had completely eliminated the need to use log tables. However the exam still contained these problems, and my student had to learn how to do them, except instead of looking up log and antilog tables, he used the log function on his calculator.

Technology, I contend, should be embraced rather than ignored.

Warm Up

The main section of this article is an investigation of a specific geometric topic (Incircles and Circumcircles) using Geometry Expressions.

As a warm up, we'll examine a handful of simple examples in which we hope to show how Geometry Expressions can be as part of a process of creative mathematics.

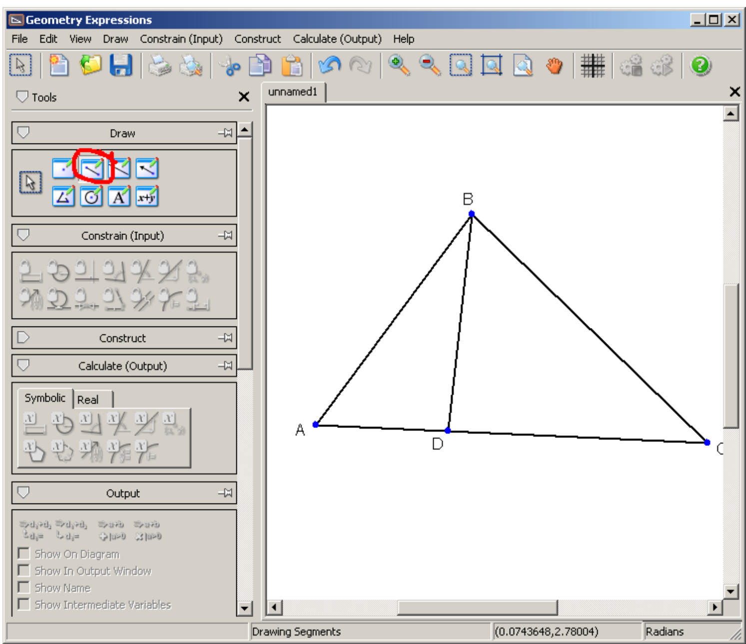
Example 1: A sequence of altitudes

We're going to look at the figure we showed in the introduction:

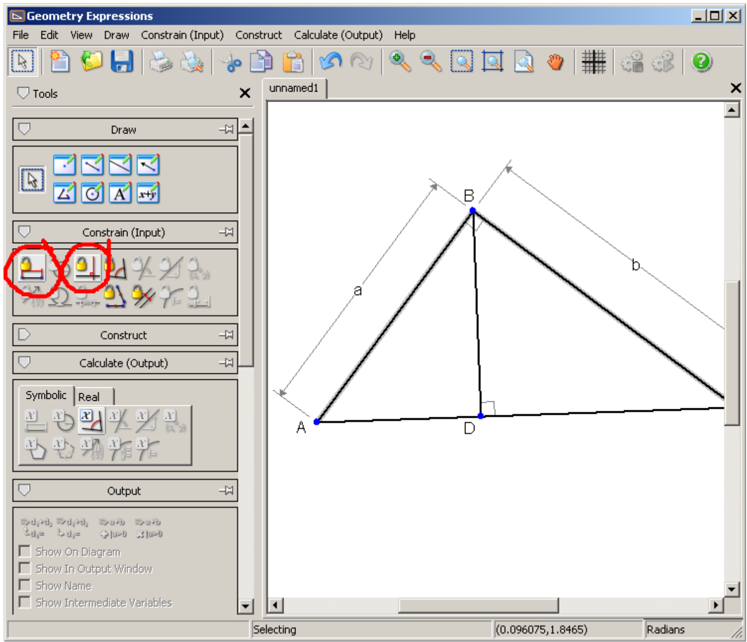
As this is Example 1, I'll show you in some detail how to create this diagram:

We will use three toolbars, the Draw toolbar to create the geometry, the Constrain toolbar to specify lengths and angles, and the Calculate toolbar to measure the length of the altitude.

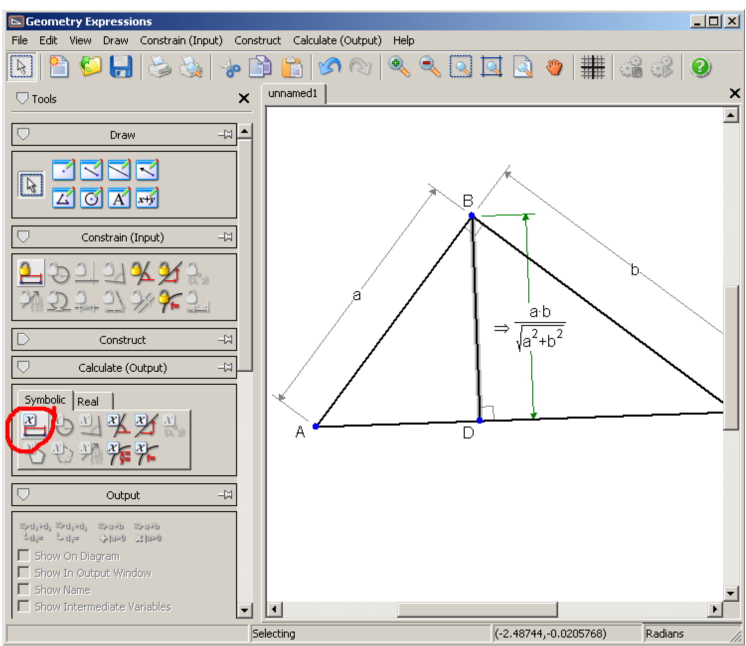
First however, we sketch the figure using the line segment tool:



Next, add the right angle constraints and the length constraints for lines AB and CD:

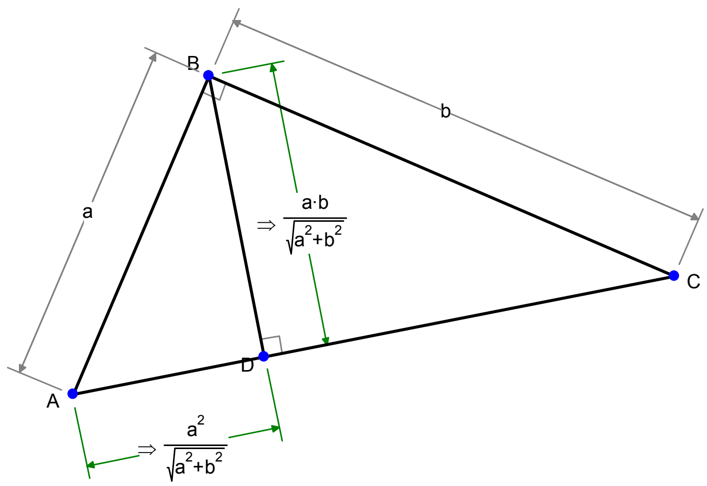


Finally, calculate the length of BD, by selecting it then clicking the Calculate Symbolic Length button:



Can you prove that the expression for the altitude is correct? (Think about the relationship between altitude and area for a triangle).

Now let's look at the length AD:

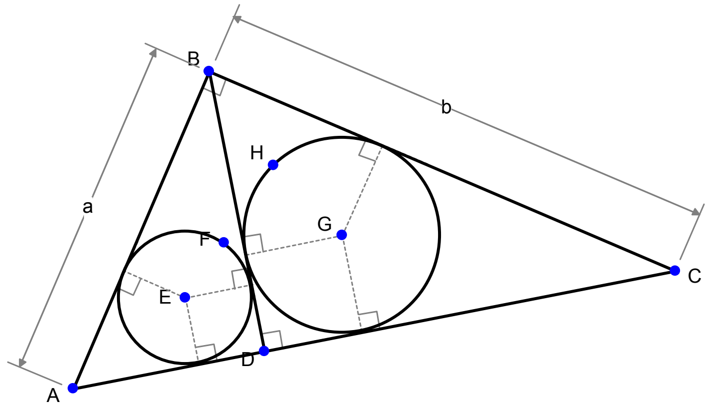


What is the ratio |AD|/|BD|? What does this tell you about the relationship between the triangle ABD and the triangle ABC? Can you establish this relationship in a different way (think angles)? Hence, can you prove the formula for AD?

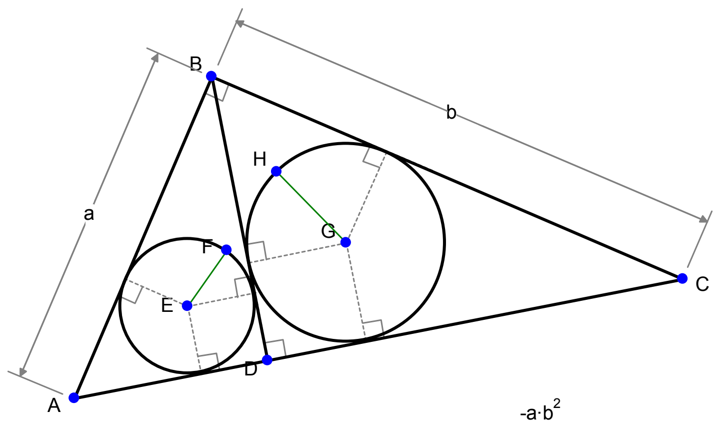
What is the length CD? What is the ratio CD/BD?

What is the relationship between triangles ABD and BCD? What is the ratio of the hypotenuse of the two triangles?

Now let's create the incircles for ABD and BCD. (To create them, sketch the circles, then apply tangent constraints between the circles and the appropriate sides of the triangles)



What is the ratio of the radii of these incircles?



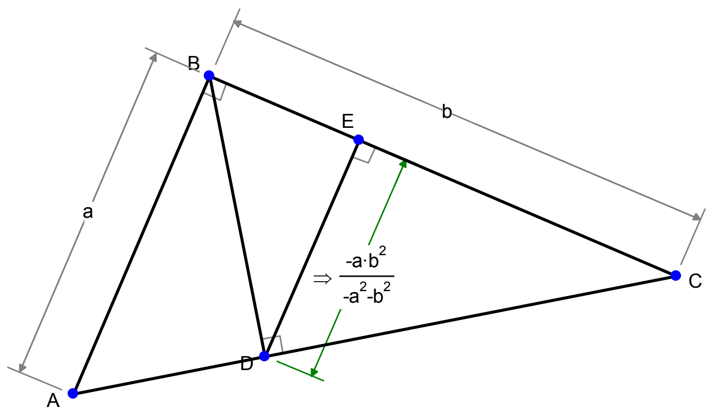
$$z_5 \Rightarrow \frac{-a\cdot b^2}{-b^2+a\cdot\left(-a-\sqrt{a^2+b^2}\right)-b\cdot\sqrt{a^2+b^2}}$$

$$z_4 \Rightarrow \frac{a^2\cdot b}{a^2+b^2+a\cdot\sqrt{a^2+b^2}+b\cdot\sqrt{a^2+b^2}}$$

$$\frac{z_4}{z_5} \Rightarrow \frac{a}{b}$$

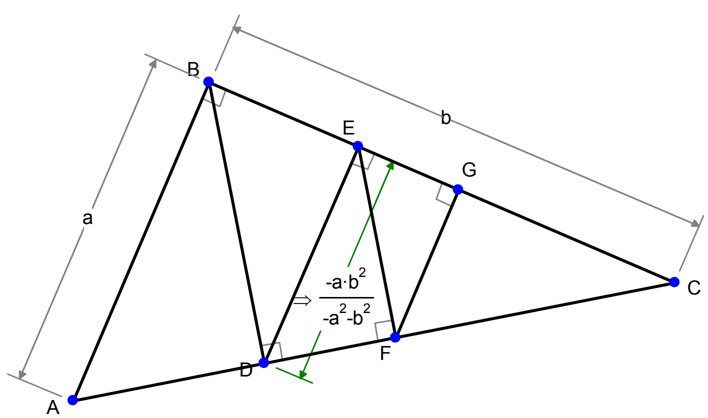
We see the ratio is a/b. Is this a surprise?

Let's go back to our original drawing and create another altitude:



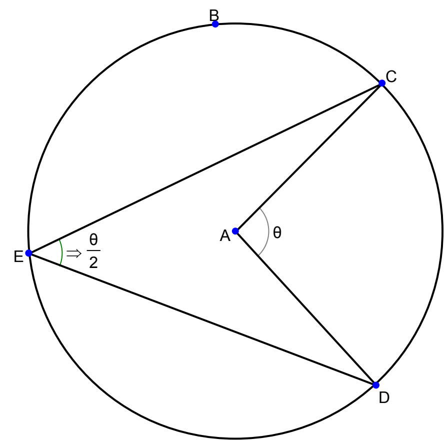
What is the ratio |DE|/|AB|? Can you prove the formula for |DE|?

Can you predict the length of FG in the drawing below?

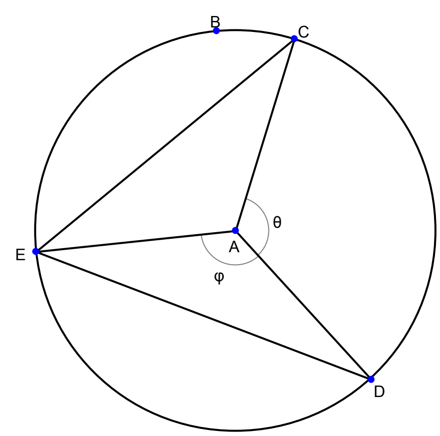


Example 2: Angles and Circles

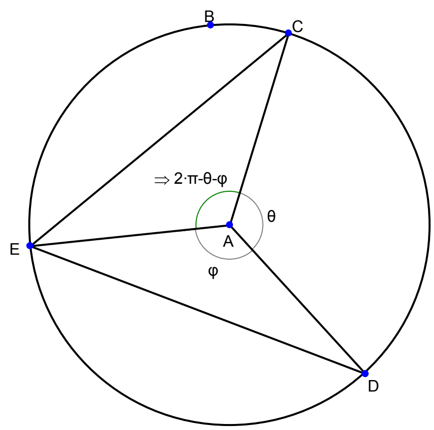
If a chord subtends an angle of θ at the center of a circle what does it subtend at the circumference?

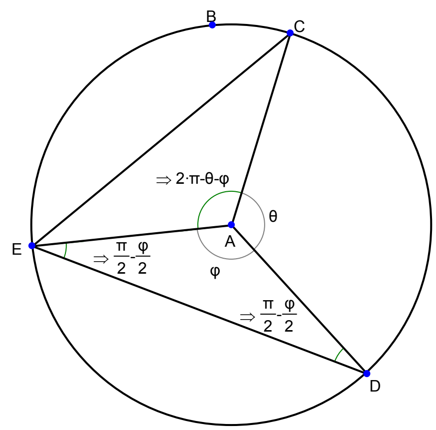


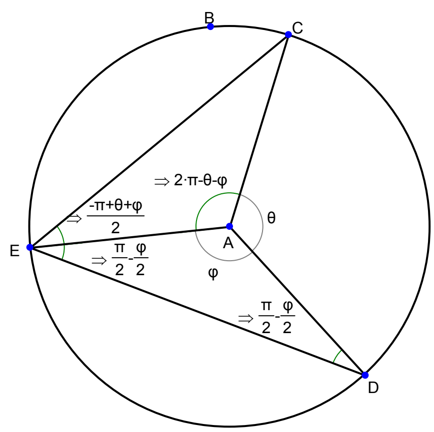
Can you prove this result. (Hint: start with the diagram below and fill in the angles).



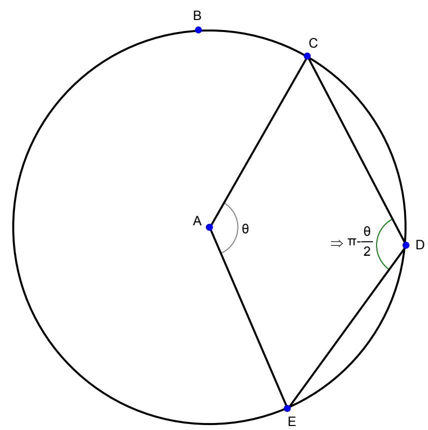
Here's a sequence of diagrams, does this constitute a proof?

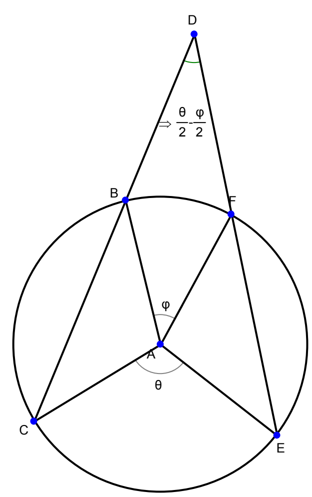






Can you follow a similar approach and prove these results:



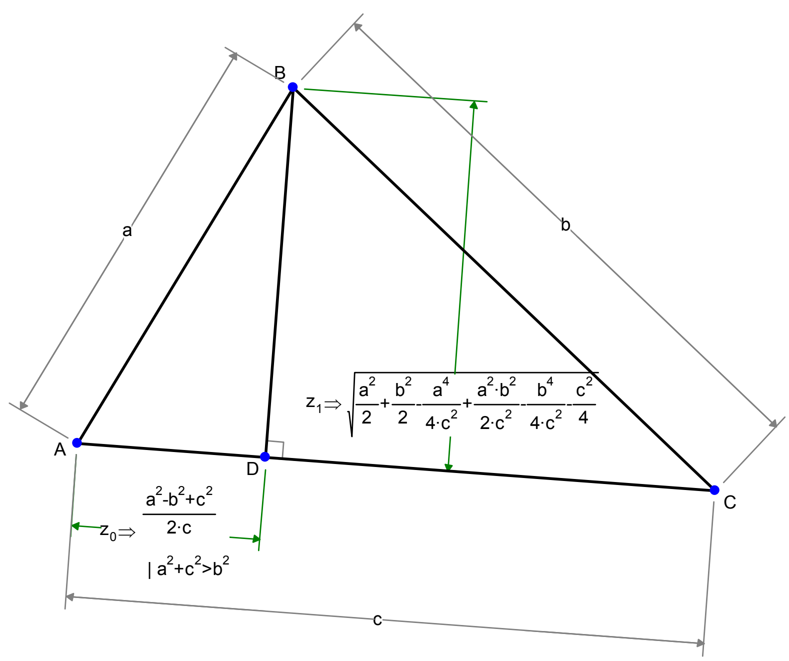


Example 3: Triangulation

Geometry Expressions has its own algebra system customized to manipulating the sort of mathematics that arises in geometry problems. However, to do further analysis of your geometry expressions you should copy them into a more fully featured algebra system. Geometry Expressions exports expressions in the MathML format, which is accepted as input by a wide variety of mathematics display and computation applications.

In this exercise, we will work on an example which involves copying into an algebra system to complete the analysis.

Assume we are writing a computer program which performs triangulation: given the length of a baseline and angles measured off the baseline to the apex of the triangle, the program is to give the coordinates of the apex. The appropriate expressions can be derived from Geometry Expressions:



In the computer program, we wish to know what the error in the derived quantities is relative to error in the measured quantities a and b.

Error in z_0 due to an error δa in a is approximately:

$$\delta z_0 \approx \delta a \frac{dz_0}{da}$$

And similarly for error in z_1

The quantity $\frac{dz_0}{da}$ can be thought of as the error magnifier.

If you select the expression z_0 you can copy and paste into an algebra system: Maple, or Mathematica or any other system which is prepared to accept mathML. We'll use the Casio ClassPad Manager. Once in there, we'll differentiate with respect to a to get an expression for the error magnification:

$$\frac{d}{da}\left(\frac{a^2+b^2\cdot(-1)+c^2}{c\cdot 2}\right)$$

$$\frac{a}{c}$$

$$\frac{d}{da}\left(\sqrt{\frac{a^2}{2}+\frac{b^2}{2}+\frac{a^4\cdot(-1)}{c^2\cdot 4}+\frac{b^2\cdot a^2}{c^2\cdot 2}+\frac{b^4\cdot(-1)}{c^2\cdot 4}+c^2\cdot\left(-\frac{1}{4}\right)}\right)$$

$$\frac{-\left(a^3-a\cdot b^2-a\cdot c^2\right)}{|c|\cdot\sqrt{-a^4-b^4-c^4+2\cdot a^2\cdot b^2+c^2\cdot\left(2\cdot a^2+2\cdot b^2\right)}}$$

The error magnifier in z_0 is simply the ratio of a/c. In z_1 the magnifier is more complicated. Factoring the term under the square root gives us a clearer picture:

$$\mathrm{factor}\left(-a^4-b^4-c^4+2\cdot a^2\cdot b^2+c^2\cdot\left(2\cdot a^2+2\cdot b^2\right)\right)$$

$$-(a+b+c)\cdot(a-b+c)\cdot(a-b-c)\cdot(a+b-c)$$

We see that the denominator of the error term goes to zero when a=b+c, or when b=a+c, or when c=a+b.

Can you interpret these conditions geometrically?

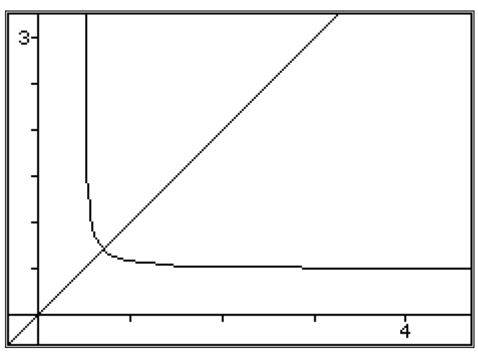
Can you interpret the complete error term geometrically? Hint: compare the denominator with the area of the triangle. Compare the numerator with the distance AD in the diagram. Can you construct a distance on the diagram whose length is more closely related to the numerator?

One question we might ask is this: what is the optimal geometry for triangulating? Let's simplify the question by assuming the triangle is isosceles and a=b=x. We'll also assume the base length is 1.

$$\frac{d}{da}\left(\frac{a^2+b^2\cdot(-1)+c^2}{c\cdot 2}\right)\Big|\{a=x,b=x,c=1\} \qquad x$$

$$\frac{d}{da}\left(\sqrt{\frac{a^2}{2}+\frac{b^2}{2}+\frac{a^4\cdot(-1)}{c^2\cdot 4}+\frac{b^2\cdot a^2}{c^2\cdot 2}+\frac{b^4\cdot(-1)}{c^2\cdot 4}+c^2\cdot\left(-\frac{1}{4}\right)}\right)\Big|\{a=x,b=x,c=1\} \qquad \frac{x}{\sqrt{4\cdot x^2-1}}$$

We can graph these functions and observe that for large values of x the first dominates, for small values of x the second dominates.



The optimal value of x is when the two are equal:

$$x$$

$$\frac{d}{da}\left(\sqrt{\frac{a^2}{2}+\frac{b^2}{2}+\frac{a^4\cdot(-1)}{c^2\cdot 4}+\frac{b^2\cdot a^2}{c^2\cdot 2}+\frac{b^4\cdot(-1)}{c^2\cdot 4}+c^2\cdot\left(-\frac{1}{4}\right)}\right)\Big|\{a=x,b=x,c=1\}$$

$$\frac{x}{\sqrt{4\cdot x^2-1}}$$

$$\text{solve}\left(x=\frac{x}{\sqrt{4\cdot x^2-1}},x\right)$$

$$\left\{x=\frac{-\sqrt{2}}{2},x=\frac{\sqrt{2}}{2}\right\}$$

What angles does this triangle have?

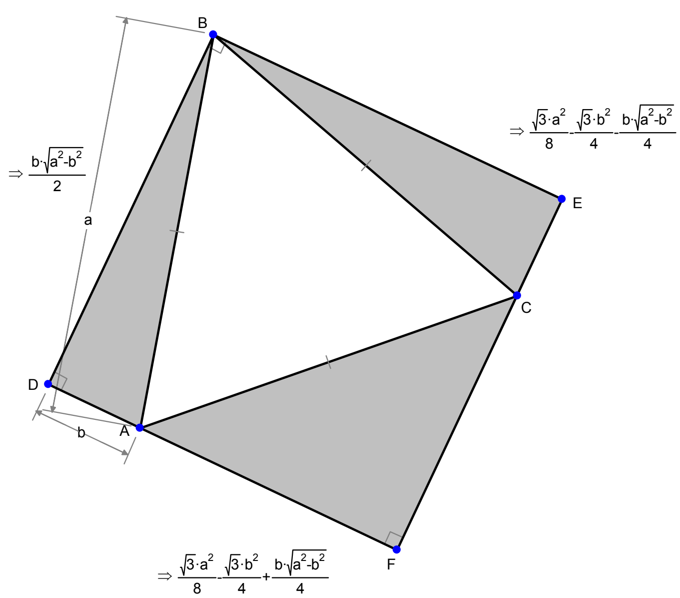
Example 4: Rectangle Circumscribing an Equilateral Triangle

Iin page 19-21 of Mathematical Gems, by Ross Honsberger (and various other places), we have the following theorem: Inscribe an equilateral triangle in a square such that one corner of the triangle is a corner of the square and the other two corners lie on the opposite sides of the square. This forms 3 right triangles. The theorem states:

> *The area of the larger right triangle is the sum of the areas of the smaller two.*

You can measure areas in Geometry Expressions by selecting a connected set of segments and constructing a polygon. You can then create an Area measure for that polygon.

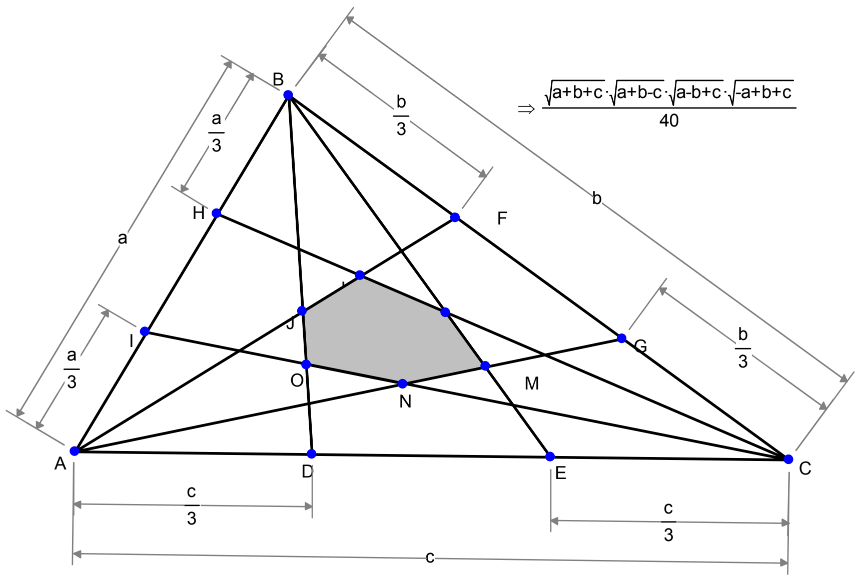
The diagram below shows the areas of the triangles in the above theorem



Can you prove the theorem from the diagram?

Can you prove that the diagram is correct?

Example 5: Area of a Hexagon bounded by Triagle side trisectors

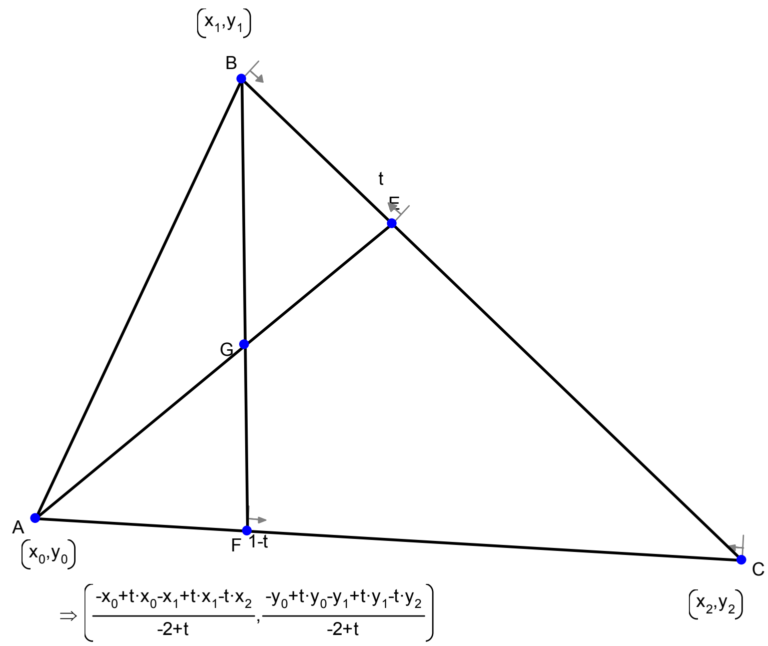


The shaded hexagon is formed by intersecting the lines joining the vertices of the triangle with the trisectors of the opposite sides.

How does the area relate to the area of the triangle ABC?

Can you prove this?

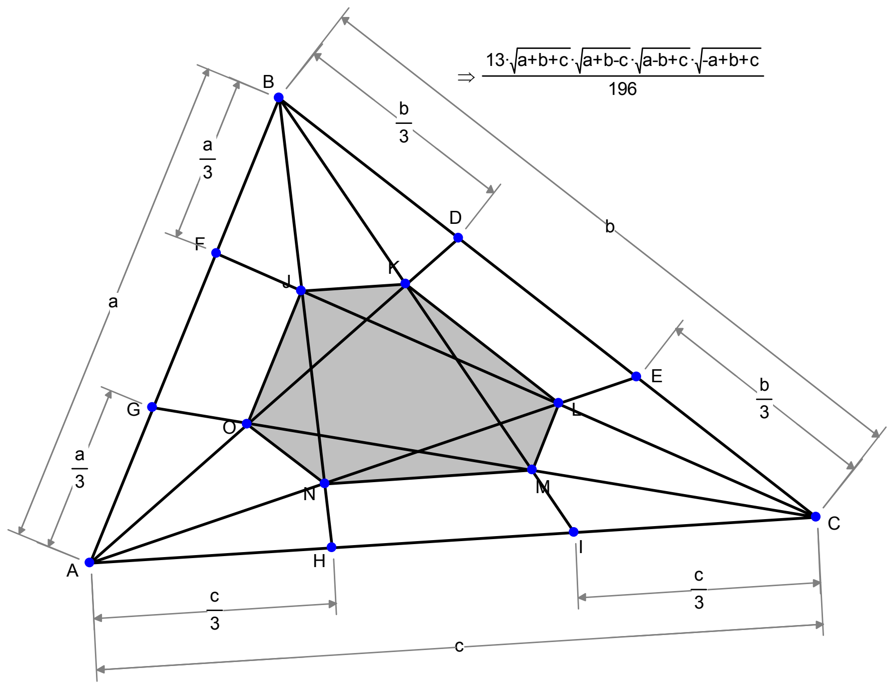
One approach uses this expression for the location of the intersection of CE and BD where E is proportion t along AB and D is proportion t along AC. Use values 1/3 and 2/3 for t to get points of the hexagon



Can you prove the area of the hexagon using this information?

Can you prove the coordinatess?

How about the area of the other hexagon?

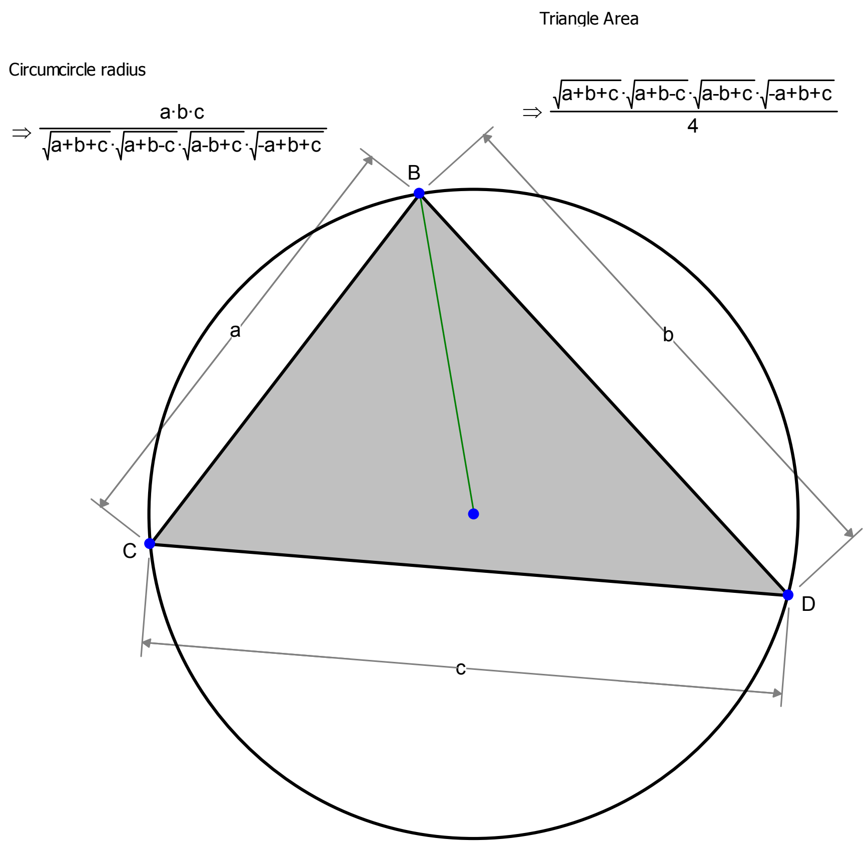


An Investigation of Incircles, Circumcircles and related matters

We'll follow with a set of examples all related to the theme of incircles, circumcircles, excircles and triangle areas.

Example 6: Circumcircle Radius

Measure the area of a triangle sides length a,b,c, and measure the radius of the circumcircle.

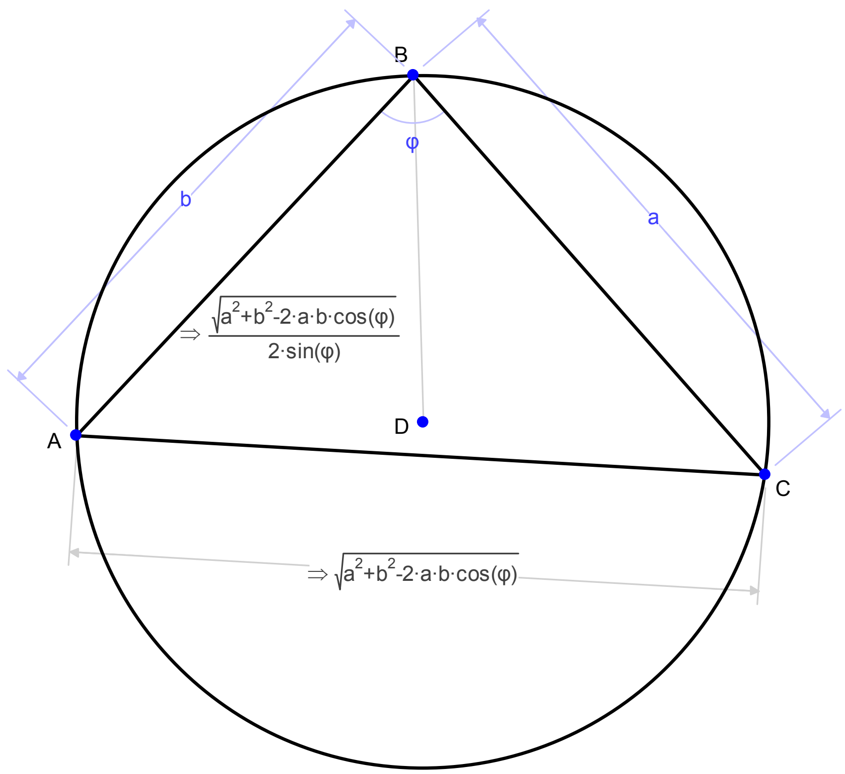


What is their relationship?

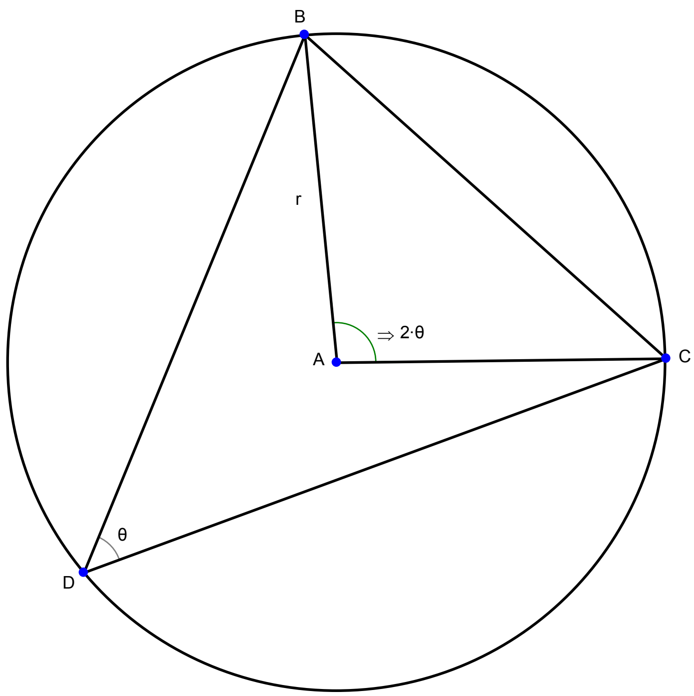
If a triangle is defined in terms of two sides and the included angle, what is its area?

Hence, derive a formula for the radius of the circumcircle which involves an angle:

You can see this is Geometry Expressions:



Can you prove that this expression is true independent of Geometry Expressions? Hint, in the diagram below, get an expression for |BC|



Let R be the radius of the circumcircle, let A,B,C be the angles of a triangle whose opposite sides have length a,b,c. At this point, you should have proved:

$$R = \frac{c}{2\sin(C)}$$

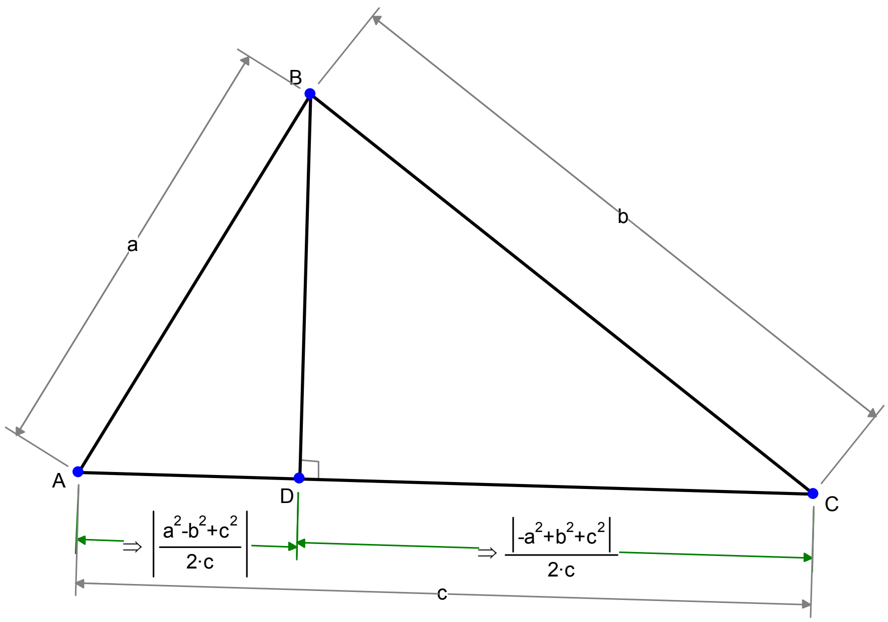
$$Area(ABC) = \frac{1}{2}ab\sin(C) = \frac{abc}{4R}$$

Hence:

$$R = \frac{abc}{4Area(ABC)}$$

Now to prove the original formula for R in terms of a,b,c, we need only prove (or accept without proof) the formula for the area of the triangle generated by Geometry Expressions.

We don't like to accept anything without proof do we?



Show that with the above values for |AD| and |CD|,

$$a^2 - |AD|^2 = b^2 - |CD|^2$$
$$|AD| + |DC| = c$$

Copy the expression for AD into your algebra system. Now create an expression for the square of the altitude. Multiply by the square of c, and divide by 4 to create an expression for the square of the area:

```
> c^2*(a^2-(1/2*(a^2-b^2+c^2)/c)^2)/4;
```

$$\frac{1}{4}c^2\left(a^2 - \frac{(a^2-b^2+c^2)^2}{4c^2}\right)$$

You can then do some algebraic manipulation to get this into a nicer form:

```
> expand(%);
```

$$\frac{1}{8}a^2c^2 - \frac{1}{16}a^4 + \frac{1}{8}a^2b^2 - \frac{1}{16}b^4 + \frac{1}{8}b^2c^2 - \frac{1}{16}c^4$$

```
> simplify(%);
```

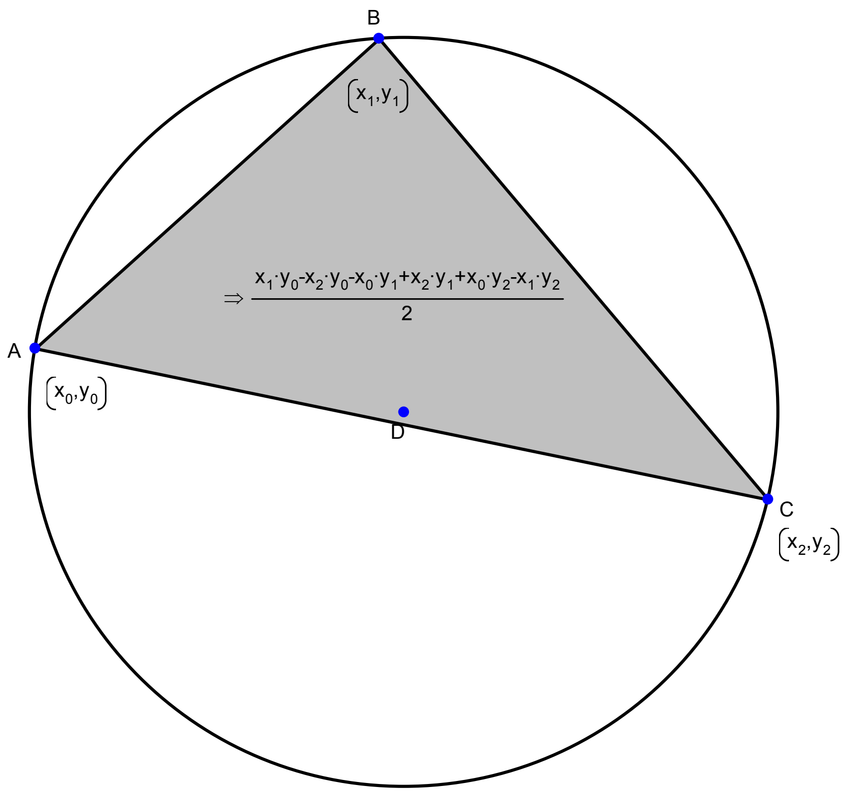
$$\frac{1}{8}a^2c^2 - \frac{1}{16}a^4 + \frac{1}{8}a^2b^2 - \frac{1}{16}b^4 + \frac{1}{8}b^2c^2 - \frac{1}{16}c^4$$

```
> factor(%);
```

$$-\frac{1}{16}(b+a+c)(b+a-c)(-c+a-b)(a-b+c)$$

As a final exercise, let’s look at the center of the circumcircle in terms of the coordinates of the triangle vertices (barycentric coordinates)

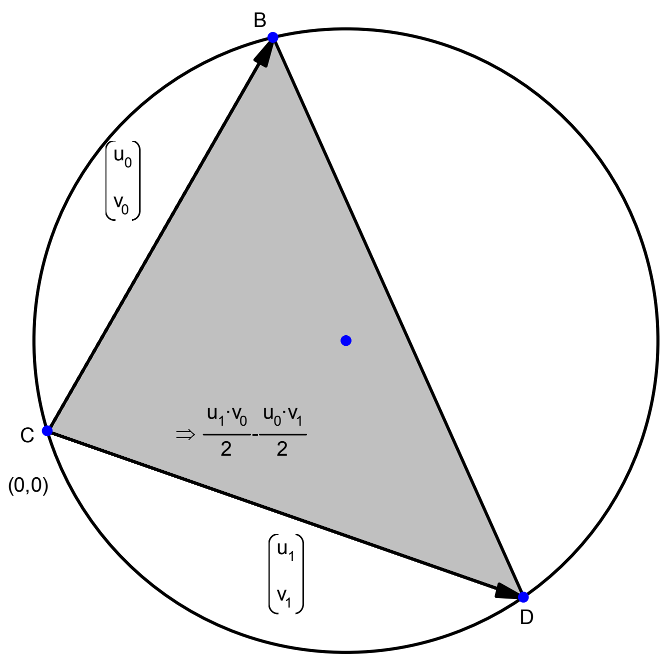
$$\Rightarrow \left(\frac{(y_1-y_2)\cdot\left(\frac{(x_0+x_1)\cdot(x_0-x_1)}{2}+\frac{(y_0+y_1)\cdot(y_0-y_1)}{2}\right)+(y_0-y_1)\cdot\left(\frac{(x_1+x_2)\cdot(-x_1+x_2)}{2}+\frac{(y_1+y_2)\cdot(-y_1+y_2)}{2}\right)}{-x_1\cdot y_0+x_2\cdot y_0+x_0\cdot y_1-x_2\cdot y_1-x_0\cdot y_2+x_1\cdot y_2}, \frac{-(x_1-x_2)\cdot\left(\frac{(x_0+x_1)\cdot}{2}\right.}{}\right.$$



The expression is quite complicated, but breaks down into constituent parts. Do you see the area embedded in the formula?

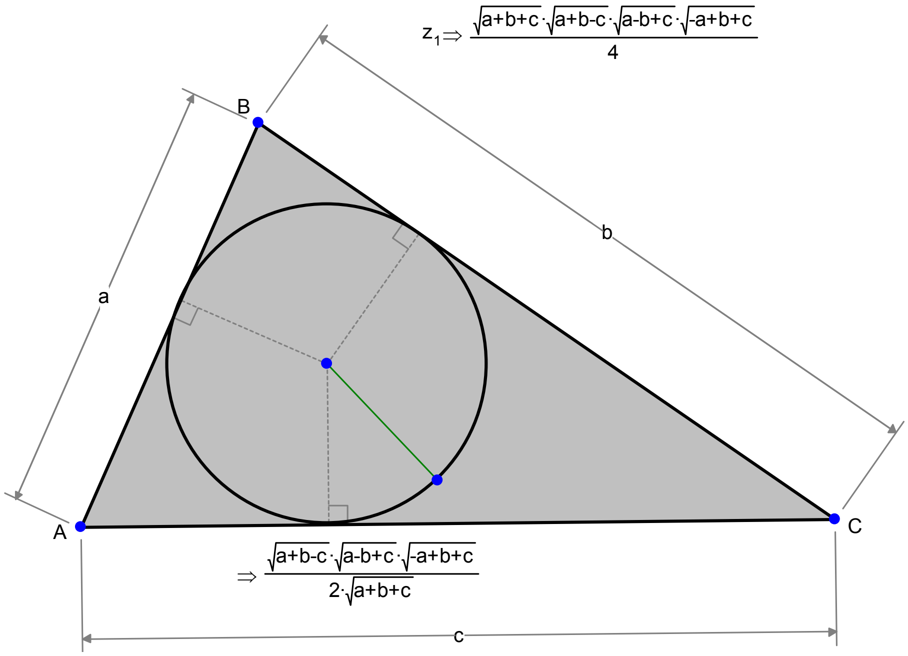
Can you write this simpler in vector terms?

$$\Rightarrow \left(\frac{-u_1^2 \cdot v_0 - v_0 \cdot v_1^2 + v_1 \cdot \left(u_0^2 + v_0^2\right)}{2 \cdot \left(-u_1 \cdot v_0 + u_0 \cdot v_1\right)}, \frac{-u_0^2 \cdot u_1 + u_0 \cdot u_1^2 - u_1 \cdot v_0^2 + u_0 \cdot v_1^2}{2 \cdot \left(-u_1 \cdot v_0 + u_0 \cdot v_1\right)} \right)$$



Example 7: Incircle Radius

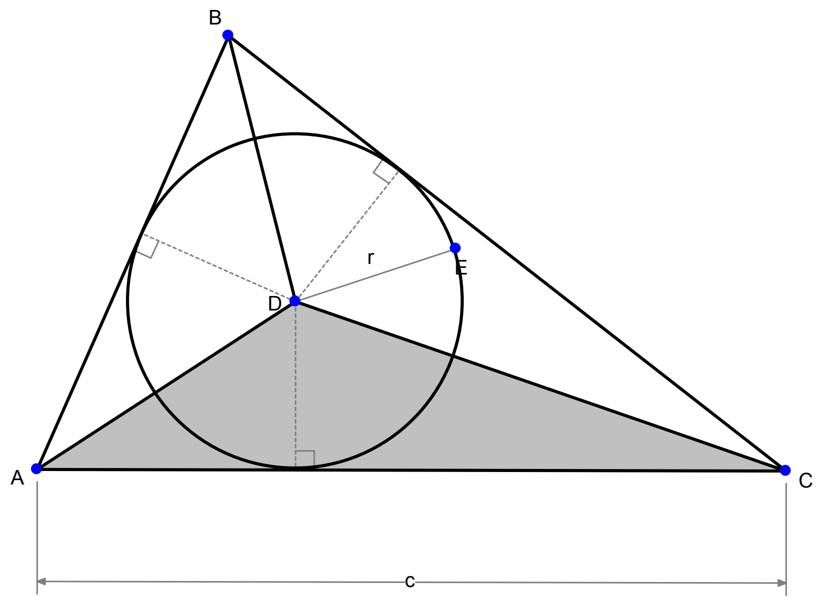
Here is the formula for the incircle radius (along with the familiar formula for the area of the triangle).



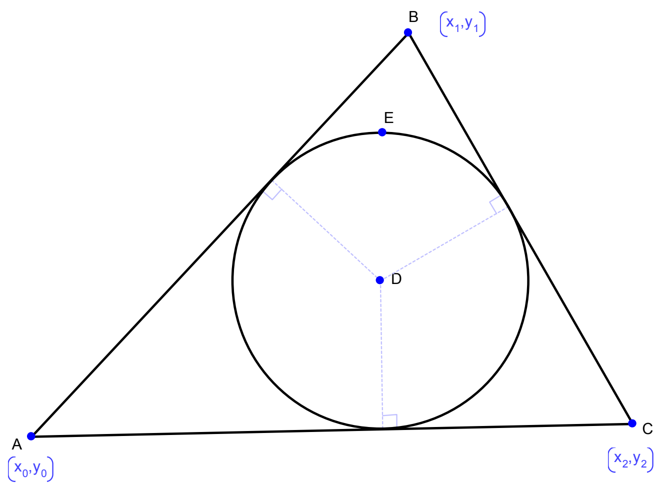
Can you express the incircle radius in terms of the area?

Now can you prove it independently of Geometry Expressions?

Hint: consider the areas of the shaded triangle:



Example 8: Incircle Center in Barycentric Coordinates

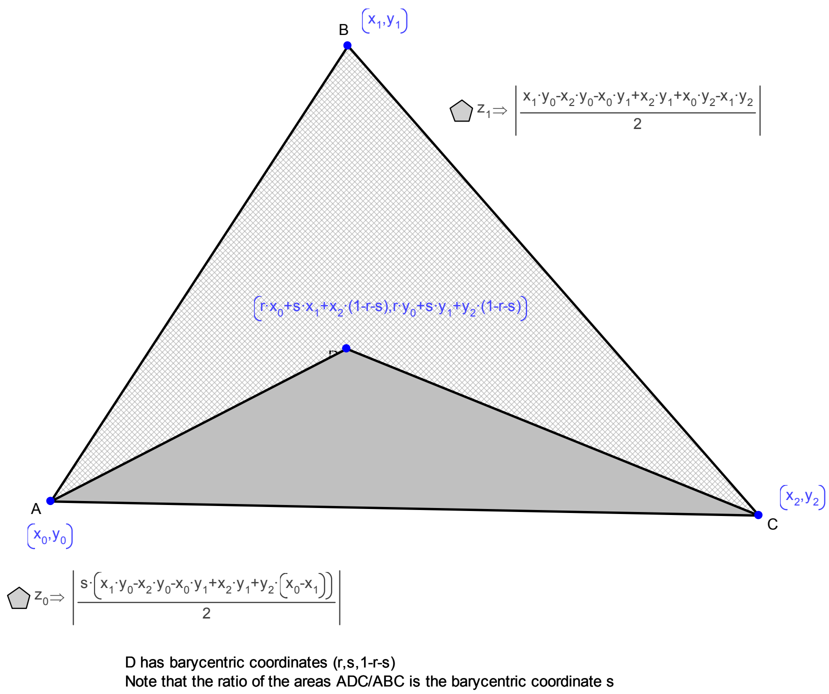


$$\Rightarrow \left(\frac{x_1 \cdot \sqrt{(x_0-x_2)^2+(-y_0+y_2)^2}+x_2 \cdot \sqrt{(-x_0+x_1)^2+(y_0-y_1)^2}+x_0 \cdot \sqrt{(-x_1+x_2)^2+(y_1-y_2)^2}}{\sqrt{(x_0-x_2)^2+(-y_0+y_2)^2}+\sqrt{(-x_0+x_1)^2+(y_0-y_1)^2}+\sqrt{(-x_1+x_2)^2+(y_1-y_2)^2}}, \frac{y_1 \cdot \sqrt{(x_0-x_2)^2+(-y_0+y_2)^2}+y_2 \cdot \sqrt{(-x_0+x_1)^2+(y_0-y_1)^2}+y_0 \cdot \sqrt{(-x_1+x_2)^2+(y_1-y_2)^2}}{\sqrt{(x_0-x_2)^2+(-y_0+y_2)^2}+\sqrt{(-x_0+x_1)^2+(y_0-y_1)^2}+\sqrt{(-x_1+x_2)^2+(y_1-y_2)^2}} \right)$$

If we let $a = |BC|$, $b=|AC|$ and $c=|AB|$ and let **A**, **B** and **C** be the position vectors of the points A,B,C then the incircle center is:

$$\frac{\mathbf{A}a+\mathbf{B}b+\mathbf{C}c}{a+b+c}$$

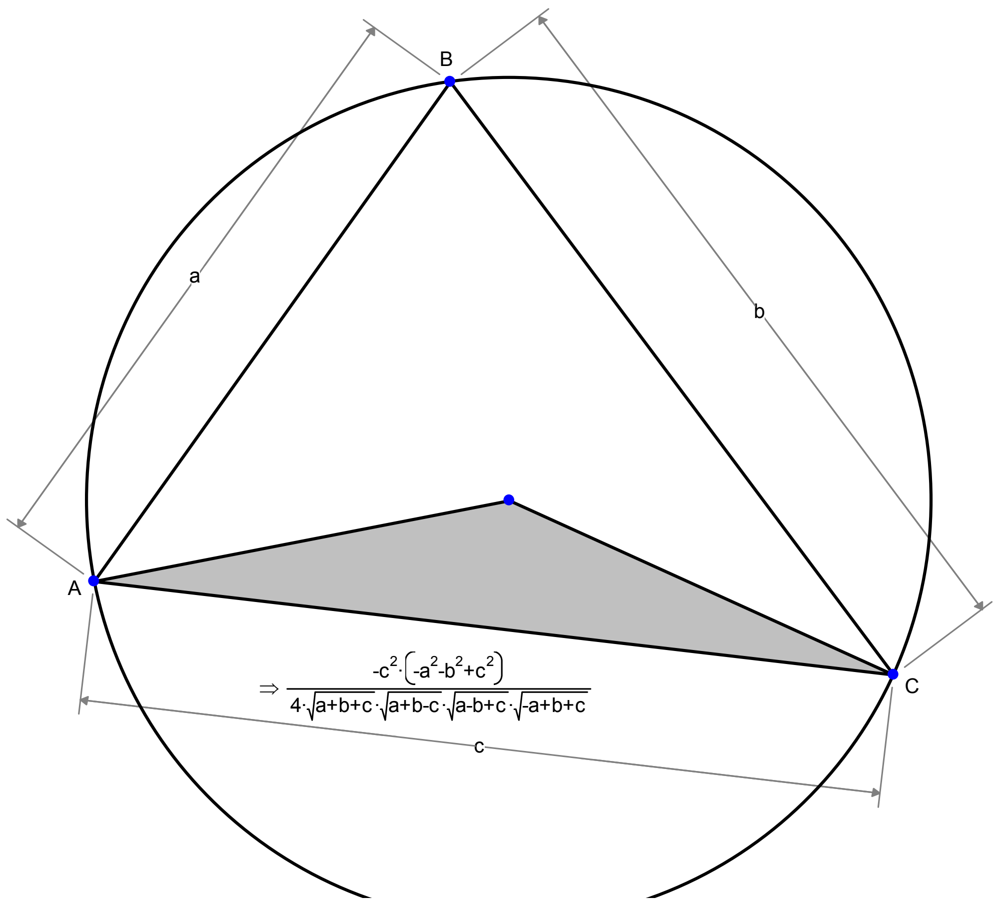
Create a point with barycentric coordinates (r,s,1-r-s) and examine the areas of the 3 triangles defined by the point and the three sides of the triangle. (you should lock the parameter values r and s to some values such that r+s<1, then the point will lie inside the triangle):



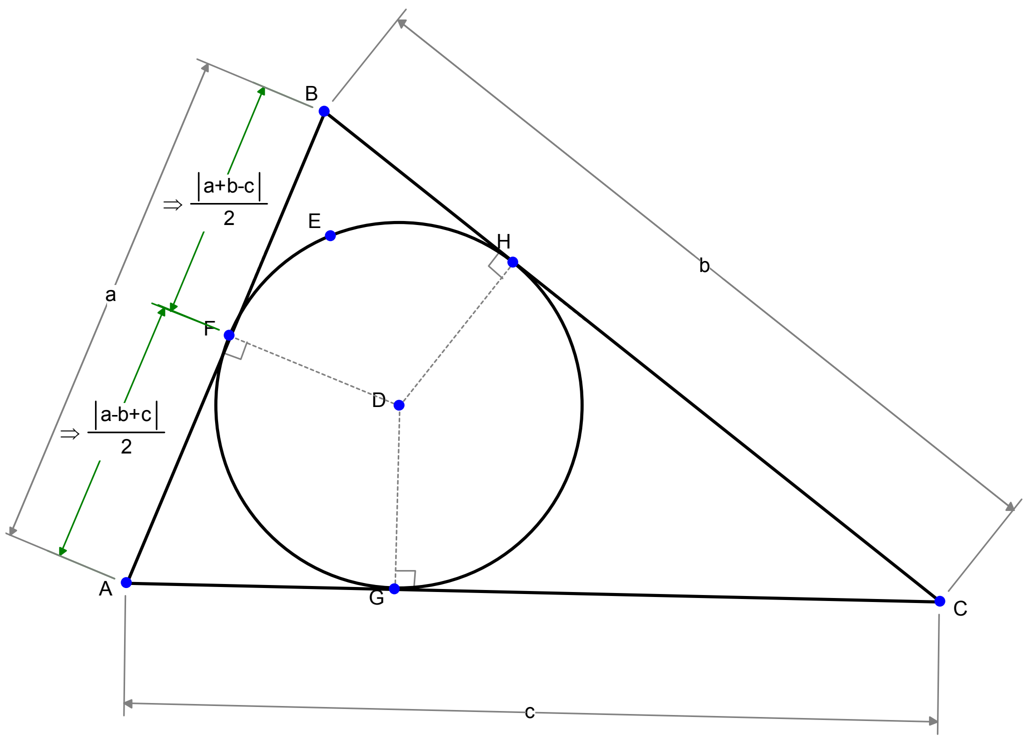
What is the ratio of this area to the area of the original triangle?

Can you use this relationship to prove the formula for the barycentric coordinates of the incenter?

Can you use this relationship to express the barycentric coordinates of the circumcenter in terms of the lengths of the sides of the triangle?



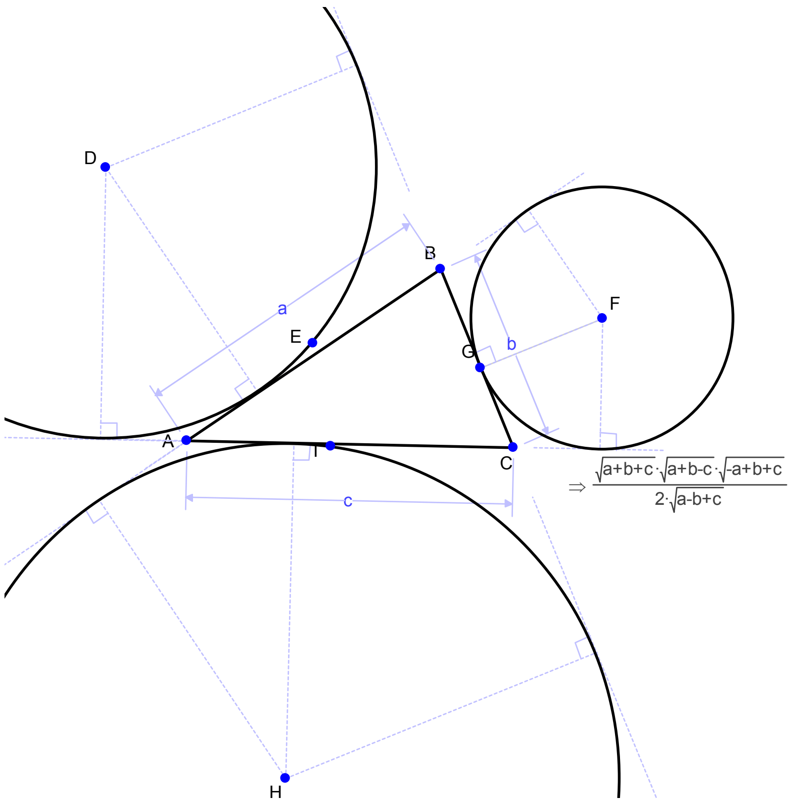
Example 9: How does the point of contact with the incircle split a line



Can you deduce the lengths BH, HC, AG, GC?

Example 10: Excircles

The three excircles of a triangle are tangent to the three sides but exterior to the circle:



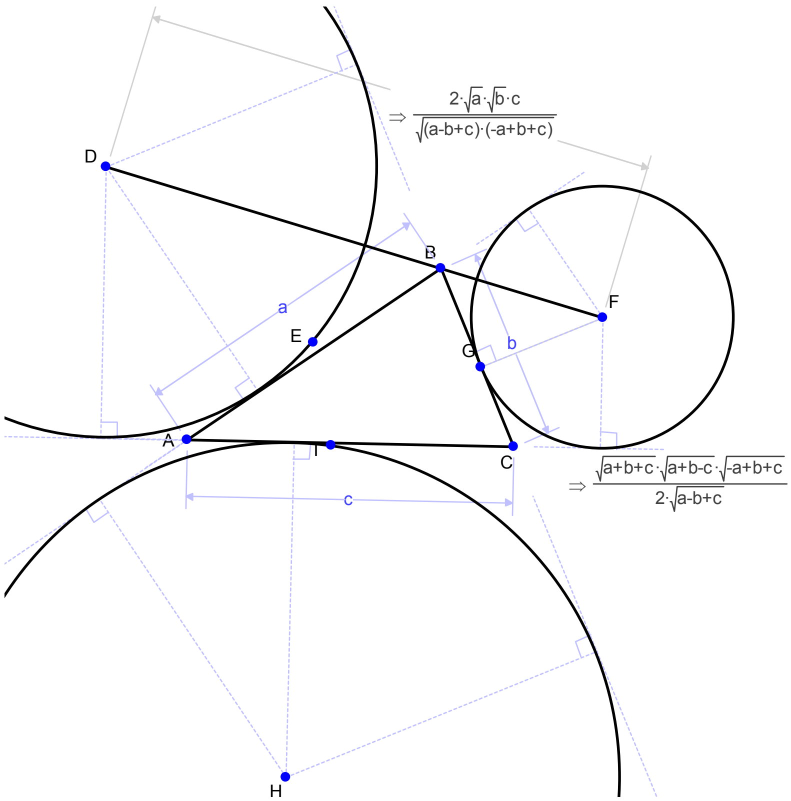
What is the ratio between this radius and the area of the triangle?

Can you prove the result using areas of triangles ABF, ACF,BCF?

What are the radii of the other two excircles?

What is the product of the radii of the three excircles and the incircle?

We examine the triangle joining the centers of the excircles:



Can you prove that B lies on DF (think in terms of symmetry)?

Can you prove the length?